

**A problem of random matrices: hermitian matrices coupled in a chain correlations and spacings**

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**I. Description of the model**

The configurations of the statistical system considered here are described by one or several  $n \times n$  matrices  $A_j$ . The statistical weight of a particular configuration is assumed to be the exponential of some energy function,  $\exp(-H)$ . The energy  $H$  contains two kinds of terms

$$H = \sum_j \text{tr} V_j(A_j) - \sum_j c_j \text{tr} (A_j A_{j+1})$$

one matrix terms  $V_j(A)$  and couplings between two matrices. The only tractable problem is that of a linear open chain of matrices with couplings between adjacent matrices, as in the above equation.

Three types of matrices are usually investigated: real symmetric, complex hermitian and quaternion self-dual. Such theoretical models are supposedly relevant to describe the physics of nuclear excitations, conductance of heterogeneous metallic wires, and even quantum gravity. The physical questions to be asked, and hopefully answered, are:

1. What is the distribution of the eigenvalues of one or several matrices ?
  2. What are the correlations between the eigenvalues of the same or different matrices ?
  3. What is the probability of finding a given number of eigenvalues in given intervals ?
- ...

From the mathematical point of view the case of complex hermitian matrices is the simplest and the above three questions can be almost fully answered, while the most relevant for physical applications is the case of real symmetric matrices.

*A common feature of all our results is that the quantities in the above questions (probability density of the eigenvalues, correlation functions and generating function of the spacings) are expressed in terms of a single determinant.*

For clarity we present below separately the cases of one matrix and of several coupled matrices. Only the coupling of matrices in an open chain is considered.

## II. One $n \times n$ hermitian matrix (summary of known results)

Probability density of the matrix elements

$$F(A) = C \exp[-\text{tr } V(A)]$$

the function  $V(A)$  depending only on the eigenvalues of  $A$ . Here and in what follows  $C$  is a generic symbol for a constant.

Eigenvalues of the matrix  $A$ :  $\mathbf{x} = \{x_1, \dots, x_n\}$ .

### 1. Probability density of the eigenvalues

$$F(\mathbf{x}) = C \Delta(\mathbf{x})^2 \exp \left[ - \sum_{i=1}^n V(x_i) \right]$$
$$\Delta(\mathbf{x}) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

### 2. Eigenvalue correlations

Let  $P_r(\xi)$  be a monic polynomial (i.e. the coefficient of its highest power is one) of order  $r$  in  $\xi$ , chosen such that

$$\int P_r(\xi) P_s(\xi) \exp[-V(\xi)] d\xi = h_r \delta_{rs}$$

and let

$$K(\xi, \eta) = \exp \left[ -\frac{1}{2} V(\xi) - \frac{1}{2} V(\eta) \right] \sum_{r=0}^{n-1} \frac{1}{h_r} P_r(\xi) P_r(\eta).$$

Then the density of an ordered set of  $k$  eigenvalues of  $A$  within small intervals around  $x_1, \dots, x_k$  is

$$R_k(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int F(\mathbf{x}) \prod_{r=k+1}^n dx_r = \det[K(x_r, x_s)]_{r,s=1,\dots,k}.$$

*For a real symmetric or a quaternion self-dual matrix  $A$ , similar formulae are known.*

### 3. Spacing functions

The spacing function  $E(k, I)$ , i.e. the probability that a chosen domain  $I$  contains exactly  $k$  eigenvalues of  $A$ , ( $0 \leq k \leq n$ ), is

$$\begin{aligned} E(k, I) &= \frac{n!}{k!(n-k)!} \int F(\mathbf{x}) \left[ \prod_{j=1}^k \chi(x_j) \right] \left[ \prod_{j=k+1}^n [1 - \chi(x_j)] \right] dx_1 \dots dx_n \\ &= \frac{1}{k!} \left( \frac{d}{dz} \right)^k R(z, I) \Big|_{z=-1} \end{aligned}$$

where  $\chi(x)$  is the characteristic function of the domain  $I$ ,  $\chi(x) = 1$ , if  $x \in I$  and  $\chi(x) = 0$ , otherwise, and  $R(z, I)$  is the generating function of the integrals over  $I$  of the correlation functions  $R_k(x_1, \dots, x_k)$ ,

$$\begin{aligned} R(z, I) &= \int F(\mathbf{x}) \prod_{j=1}^n [1 + z\chi(x_j)] dx_j = 1 + \sum_{k=1}^n \frac{\rho_k}{k!} z^k \\ \rho_k &= \int R_k(x_1, \dots, x_k) \prod_{j=1}^k \chi(x_j) dx_j. \end{aligned}$$

$R(z, I)$  can be expressed as a  $n \times n$  determinant

$$R(z, I) = \det[G_{ij}]_{i,j=0,\dots,n-1}$$

$$G_{ij} = \frac{1}{h_i} \int P_i(x) P_j(x) \exp[-V(x)] [1 + z\chi(x)] dx = \delta_{ij} + \bar{G}_{ij}$$

or as the Fredholm determinant

$$R(z, I) = \prod_{k=1}^n [1 + \lambda_k(z, I)]$$

of the integral equation

$$\int N(x, y) f(y) dy = \lambda f(x)$$

where remarkably the kernel  $N(x, y)$  is simply  $zK(x, y)\chi(y)$  with the same  $K(x, y)$  appearing in the correlation functions. The  $\lambda_i(z, I)$  are the eigenvalues of the above equation and also of the matrix  $[\bar{G}_{ij}]$ .

*Similar expressions of  $E(k, I)$  for real symmetric or quaternion self-dual matrices are known.*

### III. Case of $p$ $n \times n$ matrices coupled in an open chain

Probability density of the matrix elements

$$F(A_1, \dots, A_p) = C \exp \left[ -\text{tr} \left\{ \frac{1}{2} V_1(A_1) + V_2(A_2) + \dots + V_{p-1}(A_{p-1}) + \frac{1}{2} V_p(A_p) \right\} \right] \\ \times \exp [\text{tr} (c_1 A_1 A_2 + c_2 A_2 A_3 + \dots + c_{p-1} A_{p-1} A_p)]$$

where the  $V_j(A)$  are functions depending only on the eigenvalues of  $A$  and the  $c_j$  are real constants.

Eigenvalues of the matrix  $A_j$ :  $\mathbf{x}_j = \{x_{j1}, \dots, x_{jn}\}$ ,  $1 \leq j \leq p$ .

#### 1. Probability density of the eigenvalues

$$F(\mathbf{x}_1; \dots; \mathbf{x}_p) = C \exp \left[ -\sum_{r=1}^n \left\{ \frac{1}{2} V_1(x_{1r}) + V_2(x_{2r}) + \dots + V_{p-1}(x_{p-1r}) + \frac{1}{2} V_p(x_{pr}) \right\} \right] \\ \times \Delta(\mathbf{x}_1) \Delta(\mathbf{x}_p) \times \det [e^{c_1 x_{1r} x_{2s}}] \det [e^{c_2 x_{2r} x_{3s}}] \dots \det [e^{c_{p-1} x_{p-1r} x_{ps}}] \\ = C \Delta(\mathbf{x}_1) \Delta(\mathbf{x}_p) \left[ \prod_{k=1}^{p-1} \det [w_k(x_{kr}, x_{k+1s})]_{r,s=1, \dots, n} \right]$$

where  $\Delta(\mathbf{x}_j) = \prod_{1 \leq r < s \leq n} (x_{jr} - x_{js})$  and  $w_k(\xi, \eta) = \exp \left[ -\frac{1}{2} V_k(\xi) - \frac{1}{2} V_{k+1}(\eta) + c_k \xi \eta \right]$ .

*For real symmetric or quaternion self-dual matrices  $A_j$  similar formulae are not known.*

## 2. Eigenvalue correlations (new results)

Let  $P_r(\xi) = P_{1r}(\xi)$  and  $Q_r(\xi) = Q_{pr}(\xi)$  be monic polynomials of order  $r$  in  $\xi$ , chosen such that

$$\int P_r(\xi_1)w_1(\xi_1, \xi_2)w_2(\xi_2, \xi_3)\dots w_{p-1}(\xi_{p-1}, \xi_p)Q_s(\xi_p)d\xi_1\dots d\xi_p = h_r\delta_{rs}$$

and let

$$P_{jr}(\xi) = \int P_r(\xi_1)w_1(\xi_1, \xi_2)w_2(\xi_2, \xi_3)\dots w_{j-1}(\xi_{j-1}, \xi)d\xi_1\dots d\xi_{j-1}, \quad 1 < j \leq p$$

$$Q_{jr}(\xi) = \int w_j(\xi, \xi_{j+1})w_{j+1}(\xi_{j+1}, \xi_{j+2})\dots w_{p-1}(\xi_{p-1}, \xi_p)Q_r(\xi_p)d\xi_{j+1}\dots d\xi_p, \quad 1 \leq j < p$$

$$H_{ij}(\xi, \eta) = \sum_{r=0}^{n-1} \frac{1}{h_r} Q_{ir}(\xi)P_{jr}(\eta), \quad 1 \leq i, j \leq p$$

$$E_{ij}(\xi, \eta) = \begin{cases} \int w_i(\xi, \xi_{i+1})w_{i+1}(\xi_{i+1}, \xi_{i+2})\dots w_{j-1}(\xi_{j-1}, \eta)d\xi_{i+1}\dots d\xi_{j-1}, & 1 \leq i < j \leq p \\ 0, & i \geq j \end{cases}$$

$$K_{ij}(\xi, \eta) = H_{ij}(\xi, \eta) - E_{ij}(\xi, \eta).$$

Then the density of ordered sets of  $k_j$  eigenvalues of  $A_j$  in small intervals around  $x_{j1}, \dots, x_{jk_j}$  for  $j = 1, \dots, p$  is a  $m \times m$  determinant with  $m = k_1 + \dots + k_p$

$$\int F(\mathbf{x}_1; \dots; \mathbf{x}_p) \prod_{j=1}^p \left[ \frac{n!}{(n - k_j)!} \prod_{r=k_j+1}^n dx_{jr_j} \right] = \det [K_{ij}(x_{ir}, x_{js})]_{\substack{i, j = 1, \dots, p \\ r = 1, \dots, k_i \\ s = 1, \dots, k_j}}.$$

For real symmetric or quaternion self-dual matrices  $A_j$  similar formulae are not known.

### 3. Spacing functions (new results)

The spacing function  $E(k_1, I_1; \dots; k_p, I_p)$ , i.e. the probability that the domain  $I_j$  contains exactly  $k_j$  eigenvalues of the matrix  $A_j$  for  $j = 1, \dots, p$ ,  $0 \leq k_j \leq n$ , (the domains  $I_j$  may have overlaps), is as in the one matrix case

$$E(k_1, I_1; \dots; k_p, I_p) = \frac{1}{k_1!} \left( \frac{\partial}{\partial z_1} \right)^{k_1} \cdots \frac{1}{k_p!} \left( \frac{\partial}{\partial z_p} \right)^{k_p} R(z_1, I_1; \dots; z_p, I_p) \Big|_{z_1 = \dots = z_p = -1}$$

with the generating function

$$R(z_1, I_1; \dots; z_p, I_p) = \int F(\mathbf{x}_1; \dots; \mathbf{x}_p) \prod_{j=1}^p \prod_{r=1}^n [1 + z_j \chi_j(x_{jr})] dx_{jr}$$

$\chi_j(x)$  being the characteristic function of the domain  $I_j$ .

The generating function  $R(z_1, I_1; \dots; z_p, I_p)$  can again be expressed as a  $n \times n$  determinant

$$R(z_1, I_1; \dots; z_p, I_p) = \det[G_{ij}]_{i,j=0,\dots,n-1}$$

where

$$G_{ij} = \frac{1}{h_i} \int P_i(x_1) \left[ \prod_{k=1}^{p-1} w_k(x_k, x_{k+1}) \right] Q_j(x_p) \left[ \prod_{k=1}^p \{1 + z_k \chi_k(x_k)\} dx_k \right] = \delta_{ij} + \bar{G}_{ij}$$

or as the Fredholm determinant

$$R(z_1, I_1; \dots; z_p, I_p) = \prod_{k=1}^n [1 + \lambda_k(z_1, I_1; \dots; z_p, I_p)]$$

of an integral equation

$$\int N(x, x_p) f(x_p) dx_p = \lambda f(x).$$

Here the kernel  $N(x, y)$  is more complicated than in the one matrix case

$$N(x, x_p) = \sum_{\ell=0}^{n-1} \frac{1}{h_\ell} Q_\ell(x) \int P_\ell(x_1) \left[ \prod_{k=1}^{p-1} w_k(x_k, x_{k+1}) \right] \left[ \prod_{k=1}^p \{1 + z_k \chi_k(x_k)\} - 1 \right] \left[ \prod_{k=1}^{p-1} dx_k \right].$$

*For real symmetric or quaternion self-dual matrices  $A_j$  nothing similar is known.*

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